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Existence of Positive Solutions for Boundary Value Problems of Second-Order Functional Difference Equations

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Abstract—We prove the existence of positive solutions for boundary value problems of second-order functional difference equations. The result here is the generalization of a corresponding result for ordinary difference equations.

Keywords—Functional difference equations, Boundary value problem, Positive solution, Fixed-point index theorem in cones.

1. INTRODUCTION

In a recent paper [1], Jiang and Weng studied the existence of positive solutions of the following boundary value problems of second-order functional differential equations of the form

$$\begin{aligned} y''(x) + r(x) f(y(w(x))) &= 0, && 0 < x < 1, \\ \alpha y(x) - \beta y'(x) &= \varsigma(x), && a \le x \le 0, \\ \gamma y(x) + \delta y'(x) &= \eta(x), && 1 \le x \le b. \end{aligned} \tag{1.1$'$}$$

In this article, a discrete analogue of the BVP $(1.1)'$ is considered. We investigate the existence of positive solutions for the boundary value problems of a second-order functional difference equation (FDE) of the form

$$\begin{aligned} -\Delta^2 y(n-1) &= f(n, y(w(n))), && n \in [a,b], \\ \alpha y(n-1) - \beta \Delta y(n-1) &= \varsigma(n), && n \in [\tau_1, a], \\ \gamma y(n) + \delta \Delta y(n) &= \eta(n), && n \in [b, \tau_2], \end{aligned} \tag{1.1}$$

where a, b $(b > a+1)$ are integers and $[a,b]$ denote the discrete set $\{a, a+1, \dots, b\}$. As usual, Δ denotes the forward difference operator defined by

$$\Delta y(n) = y(n+1) - y(n).$$

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We will assume that the following conditions are satisfied.

(H1) $w(n)$ is defined on $[a,b]$ satisfies

$$c = \inf\{w(n) : a \le n \le b\} < b,$$
$$d = \sup\{w(n) : a \le n \le b\} > a.$$

Let $E_1 := \{n \in E : a \le w(n) \le b-1\}$ be nonempty subset of

$$E := \{n \in [a,b] : a \le w(n) \le b\}.$$

(H2) $\alpha, \gamma, \delta \ge 0$, $\beta > 0$, $\gamma + \delta > 0$, $\alpha + \gamma > 0$.

(H3) $\varsigma(n)$ and $\eta(n)$ are defined on $[\tau_1 + 1, a]$ and $[b, \tau_2]$ respectively, where $\tau_1 := \min\{a, c\}$, $\tau_2 := \max\{b, d\}$; furthermore, $\varsigma(a) = \eta(b) = 0$;

$$\sum_{k=n+1}^{a} \varsigma(k)\left(1+\frac{\alpha}{\beta}\right)^{-k} \ge 0, \qquad \text{for } n \in [\tau_1, a];$$
$$\eta(n) \ge 0, \qquad \text{for } n \in [b, \tau_2] \text{ as } \delta = 0;$$
$$\left(1-\frac{\gamma}{\delta}\right)^{n}\sum_{k=b}^{n-1} \eta(k)\left(1-\frac{\gamma}{\delta}\right)^{-(k+1)} \ge 0, \qquad \text{for } n \in [b, \tau_2] \text{ as } \delta > 0.$$

(H4) $f : [a,b] \times \mathbf{R} \to \mathbf{R}$ is continuous with respect to y and $f(n,y) \ge 0$ for $y \in \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the set of nonnegative real numbers.

For the case of $w(n) \equiv n$, the BVP (1.1) becomes the two-point BVP of ODE.

Boundary value problems for functional difference equations have been studied in recent years. We cite some appropriate references here [2,3]. Reid and Yin [2] applied a cone theoretic fixed-point theorem to obtain a λ interval on which there exists a nontrivial positive solution of the functional difference equation,

$$-\Delta^2 x(t-1) + \lambda a(t) f(x(\phi(t))) = 0, \qquad t \in [1, T+1], \tag{1.2}$$

subject to the initial value

$$x(j) = v(j), \qquad j \in [-r, 0], \tag{1.3}$$

and the boundary conditions

$$x(0) = 0, \qquad x(T+2) = 0. \tag{1.4}$$

Eloe, *et al.* [3] applied a cone theoretic fixed-point theorem and obtained sufficient conditions for the existence of positive solutions of a boundary value problem for the functional difference equations,

$$-\Delta^2 x(t-1) + \lambda a(t) f(x(t), x(\phi(t))) = 0, \qquad t \in [1, T+1]. \tag{1.5}$$

The solution x is required to satisfy the initial value

$$x(j) = v(j), \qquad j \in [-r, -1], \tag{1.6}$$

and the boundary conditions (1.4). They considered analogues of sublinear or superlinear growth in the nonlinear terms.

In this paper, we shall use a fixed-point index theorem in cones to investigate the existence of positive solutions to BVP (1.1). For the special case $\lambda = 1$, our problem is more general than the problems (1.2)–(1.4). Besides, our results include the situations that f is either superlinear or sublinear.

2. MAIN THEOREMS

First, we give the following definitions of solution and positive solution of the BVP (1.1). Then, we give lemmas which will be used later.

DEFINITION. *$y(n)$ is said to be a solution of BVP (1.1), if it satisfies the following:*

1. *$y(n)$ is nonnegative;*
2. *$y(n) = y(\tau_1; n)$ as $n \in [\tau_1, a]$, where $y(\tau_1; n) : [\tau_1, a] \to [0, +\infty)$ is defined by*

$$y(\tau_1; n) = \left(1+\frac{\alpha}{\beta}\right)^n \left[\frac{1}{\beta} \sum_{k=n+1}^{a} \varsigma(k) \left(1+\frac{\alpha}{\beta}\right)^{-k} + \left(1+\frac{\alpha}{\beta}\right)^{-a} y(a)\right]; \tag{2.1}$$

3. *$y(n) = y(\tau_2; n)$ as $n \in [b, \tau_2]$, where $y(\tau_2; n) : [b, \tau_2] \to [0, +\infty)$ is defined by*

$$y(\tau_2; n) = \begin{cases} \left(1-\frac{\gamma}{\delta}\right)^n \left[\frac{1}{\delta} \sum\limits_{k=b}^{n-1} \eta(k) \left(1-\frac{\gamma}{\delta}\right)^{-(k+1)} + \left(1-\frac{\gamma}{\delta}\right)^{-b} y(b)\right], & \delta > 0, \\ \dfrac{1}{\gamma} \eta(n), & \delta = 0. \end{cases} \tag{2.2}$$

Denote by $\varphi(n)$ and $\psi(n)$, the solutions of the corresponding homogeneous equation

$$-\Delta^2 y(n-1) = 0, \qquad n \in [a, b], \tag{2.3}$$

under the initial conditions

$$\begin{aligned} \varphi(a-1) &= \beta, & \Delta\varphi(a-1) &= \alpha; \\ \psi(b) &= \delta, & \Delta\psi(b) &= -\gamma. \end{aligned} \tag{2.4}$$

Define the number D by

$$D := \alpha\psi(a-1) - \beta\Delta\psi(a-1) = \gamma\varphi(b) + \delta\Delta\varphi(b). \tag{2.5}$$

Using the initial conditions (2.4), we can deduce from equation (2.3) for $\varphi(n)$ and $\psi(n)$, the following equations:

$$\varphi(n) = \beta + \alpha(n-a+1), \tag{2.6}$$

$$\psi(n) = \delta + \gamma(b-n). \tag{2.7}$$

(See [4].) Let $G(n,s)$ be the Green's function for the boundary value problem

$$\begin{aligned} -\Delta^2 y(n-1) &= 0, \qquad n \in [a, b], \\ \alpha y(a-1) - \beta\Delta y(a-1) &= 0, \\ \gamma y(b) + \delta\Delta y(b) &= 0, \end{aligned}$$

is given by

$$G(n,s) := \frac{1}{D} \begin{cases} \varphi(n)\psi(s), & a-1 \le n \le s \le b+1, \\ \varphi(s)\psi(n), & a-1 \le s \le n \le b+1, \end{cases} \tag{2.8}$$

where $\varphi(n)$ and $\psi(n)$ are given in (2.6) and (2.7), respectively, and $D := \alpha\delta + \beta\gamma + \alpha\gamma(b-a+1)$. It is obvious from (H2), that $D > 0$ holds.

Suppose that $y(n)$ is a solution of the BVP (1.1), then, it could be expressed as

$$y(n) = \begin{cases} y(\tau_1; n), & \tau_1 \le n \le a, \\ \sum\limits_{s=a}^{b} G(n,s) f(s, y(w(s))), & a \le n \le b, \\ y(\tau_2; n), & b \le n \le \tau_2. \end{cases} \tag{2.9}$$

Furthermore, a solution $y(n)$ of (1.1) is called a positive solution, if $y(n) > 0$, for $n \in [a, b]$.

LEMMA 2.1. *Assume that Condition (H2) is satisfied. Then,*

(i) $0 \le G(n,s) \le G(s,s)$, *for* $n, s \in [a-1, b]$;

(ii) $G(n,s) \ge \sigma G(s,s)$, *for* $n \in [a, b-1]$ *and* $s \in [a-1, b]$,

where

$$\sigma = \min\left\{\frac{\beta+\alpha}{\beta+\alpha(b-a+1)}, \frac{\delta+\gamma}{\delta+\gamma(b-a+1)}\right\}. \tag{2.10}$$

PROOF. $\varphi(n) > 0$, for $n \in [a-1, b+1]$, and $\psi(n) \ge 0$, for $n \in [a-1, b]$. Besides, $\varphi(n)$ is nondecreasing and $\psi(n)$ is nonincreasing, for $n \in [a-1, b+1]$. Therefore, we have, for $a-1 \le n \le s \le b$,

$$G(n,s) = \frac{1}{D}\varphi(n)\psi(s) \le \frac{1}{D}\varphi(s)\psi(s) = G(s,s),$$

and we have, for $a-1 \le s \le n \le b$,

$$G(n,s) = \frac{1}{D}\varphi(s)\psi(n) \le \frac{1}{D}\varphi(s)\psi(s) = G(s,s).$$

So, Statement (i) of the lemma is proved. ∎

If $G(s,s) = 0$, for a given $s \in [a-1, b]$, then Statement (ii) of the lemma is obvious for such values. Now, let $s \in [a-1, b]$ and $G(s,s) \ne 0$. Consequently, $G(s,s) > 0$, for all such s. Let us take any $n \in [a, b-1]$. Then, we have, for $s \in [a-1, n]$,

$$\frac{G(n,s)}{G(s,s)} = \frac{\psi(n)}{\psi(s)} \ge \frac{\psi(b-1)}{\psi(a-1)} = \frac{\delta+\gamma}{\delta+\gamma(b-a+1)},$$

and we have, for $s \in [n, b]$,

$$\frac{G(n,s)}{G(s,s)} = \frac{\varphi(n)}{\varphi(s)} \ge \frac{\varphi(a)}{\varphi(b)} = \frac{\beta+\alpha}{\beta+\alpha(b-a+1)}.$$

Note that the number σ defined by (2.10) satisfies the inequalities $0 < \sigma < 1$.

LEMMA 2.2. *(See [5–8].) Assume that $\mathcal{B}$ is a Banach space, and $\mathcal{K} \subset \mathcal{B}$ is a cone in $\mathcal{B}$. Let $\mathcal{K}_p = \{u \in \mathcal{K} : \|u\| < p\}$. Furthermore, assume that $\Phi : \mathcal{K} \to \mathcal{K}$ is a compact map and $\Phi u \ne u$ for $u \in \partial\mathcal{K}_p = \{u \in \mathcal{K} : \|u\| = p\}$. Then, one has the following conclusions.*

1. *If $\|u\| \le \|\Phi u\|$, for $u \in \partial\mathcal{K}_p$, then, $i(\Phi, \mathcal{K}_p, \mathcal{K}) = 0$.*
2. *If $\|u\| \ge \|\Phi u\|$, for $u \in \partial\mathcal{K}_p$, then, $i(\Phi, \mathcal{K}_p, \mathcal{K}) = 1$.*

We shall show the conclusion of Theorems 2.1 and 2.2 only for the situation $\delta > 0$. The arguments for $\delta = 0$ are similar. Throughout this paper, we assume that $u_0(n)$ is the solution of (BVP) with $f \equiv 0$. Clearly, it can be expressed as follows.

$$u_0(n) = \begin{cases} \left(1+\dfrac{\alpha}{\beta}\right)^n \left[\dfrac{1}{\beta}\sum\limits_{k=n+1}^{a} \varsigma(k)\left(1+\dfrac{\alpha}{\beta}\right)^{-k}\right], & \tau_1 \le n \le a, \\ 0, & a \le n \le b, \\ \left(1-\dfrac{\gamma}{\delta}\right)^n \left[\dfrac{1}{\delta}\sum\limits_{k=b}^{n-1} \eta(k)\left(1-\dfrac{\gamma}{\delta}\right)^{-(k+1)}\right], & b \le n \le \tau_2. \end{cases} \tag{2.11}$$

Let $y(n)$ be a solution of (BVP) and $u(n) = y(n) - u_0(n)$. Noting that $u(n) \equiv y(n)$, for $n \in [a, b]$, then, by using (2.11), we have

$$u(n) = \begin{cases} \left(1+\dfrac{\alpha}{\beta}\right)^{n-a} u(a), & \tau_1 \le n \le a, \\ \sum\limits_{s=a}^{b} G(n,s) f(s, u(w(s)) + u_0(w(s))), & a \le n \le b, \\ \left(1-\dfrac{\gamma}{\delta}\right)^{n-b} u(b), & b \le n \le \tau_2. \end{cases} \tag{2.12}$$

Let $\mathcal{K}$ be a cone in the $\tau_2 - \tau_1 + 1$-dimensional real Banach space $\mathcal{B}$ of real-valued functions $u(n)$ defined on $[\tau_1, \tau_2]$ by $\mathcal{K} = \{u \in \mathcal{B} : \min_{a \le n \le b-1} u(n) \ge \sigma \|u\|\}$, where $\|u\| := \max_{\tau_1 \le n \le \tau_2} |u(n)|$ and σ is defined by (2.10).

Define an operator $\Phi : \mathcal{K} \to \mathcal{K}$ by

$$(\Phi u)(n) = \begin{cases} \left(1 + \dfrac{\alpha}{\beta}\right)^{n-a} \sum\limits_{s=a}^{b} G(a,s) f(s, u(w(s)) + u_0(w(s))), & \tau_1 \le n \le a, \\ \sum\limits_{s=a}^{b} G(n,s) f(s, u(w(s)) + u_0(w(s))), & a \le n \le b, \\ \left(1 - \dfrac{\gamma}{\delta}\right)^{n-b} \sum\limits_{s=a}^{b} G(b,s) f(s, u(w(s)) + u_0(w(s))), & b \le n \le \tau_2. \end{cases} \tag{2.13}$$

LEMMA 2.3. $\Phi(\mathcal{K}) \subset \mathcal{K}$.

PROOF. For $\tau_1 \le n \le a$ and $b \le n \le \tau_2$, we have $0 \le (\Phi u)(n) \le (\Phi u)(a)$ and $0 \le (\Phi u)(n) \le (\Phi u)(b)$, respectively. Thus, $\|\Phi u\|_{[\tau_1,\tau_2]} = \|\Phi u\|_{[a,b]}$.

It follows from the definition of $\mathcal{K}$ and Lemma 2.1, that

$$\begin{aligned} \min_{a \le n \le b-1} (\phi u)(n) &= \min_{a \le n \le b-1} \sum_{s=a}^{b} G(n,s) f(s, u(w(s)) + u_0(w(s))) \\ &\ge \sigma \sum_{s=a}^{b} G(s,s) f(s, u(w(s)) + u_0(w(s))) \\ &\ge \sigma \max_{a \le n \le b} \sum_{s=a}^{b} G(n,s) f(s, u(w(s)) + u_0(w(s))) \\ &= \sigma \|\Phi u\|_{[a,b]} = \sigma \|\Phi u\|_{[\tau_1,\tau_2]}, \end{aligned}$$

which implies $\Phi(\mathcal{K}) \subset \mathcal{K}$.

LEMMA 2.4. *$\Phi : \mathcal{K} \to \mathcal{K}$ is completely continuous.*

LEMMA 2.5. *If*

$$\lim_{v \to 0^+} \frac{f(n,v)}{v} = \infty \qquad \text{and} \qquad \lim_{v \to +\infty} \frac{f(n,v)}{v} = \infty, \tag{2.14}$$

for all $n \in [a,b]$, then, there exist $0 < r_0 < R_0 < \infty$, such that $i(\Phi, \mathcal{K}_r, \mathcal{K}) = 0$, for $0 < r \le r_0$ and $i(\Phi, \mathcal{K}_R, \mathcal{K}) = 0$, for $R \ge R_0$.

PROOF. Choose $M > 0$, such that

$$\sigma^2 M \sum_{E_1} G(s,s) > 1. \tag{2.15}$$

By using the first equality of (2.14), we can choose $r_0 > 0$, such that

$$f(n,v) \ge Mv, \qquad 0 \le v \le r_0,$$

If $u \in \partial\mathcal{K}_r$ $(0 < r \le r_0)$, then, for $n_0 \in [a, b-1]$, we obtain (noting that $u_0(n) \equiv 0$ as $n \in [a,b]$),

$$\begin{aligned} (\Phi u)(n_0) &= \sum_{s=a}^{b} G(n_0,s) f(s, u(w(s)) + u_0(w(s))) \\ &\ge \sigma \sum_{s=a}^{b} G(s,s) f(s, u(w(s)) + u_0(w(s))) \\ &\ge \sigma \sum_{E_1} G(s,s) f(s, u(w(s))) \end{aligned}$$

$$\begin{aligned}&\geq \sigma M \sum_{E_1} G(s,s)\, u(w(s)) \\ &\geq \sigma^2 \|u\| M \sum_{E_1} G(s,s) \\ &> \|u\|.\end{aligned}$$

This leads to

$$\|\Phi u\| > \|u\|, \qquad \forall u \in \partial\mathcal{K}_r.$$

Thus, we have from Lemma 2.2, $i(\Phi, \mathcal{K}_r, \mathcal{K}) = 0$, for $0 < r \leq r_0$. On the other hand, the second equality of (2.14) implies that for every $M > 0$, there is an $R_0 > r_0$, such that

$$f(n,v) \geq Mv, \qquad v \geq \sigma R_0. \tag{2.16}$$

Here, we choose $M > 0$, satisfying (2.15). For $u \in \partial\mathcal{K}_R$, $R \geq R_0$, we have from the definition of $\mathcal{K}_R$ that

$$u(n) \geq \sigma \|u\| = \sigma R, \qquad n \in [a, b-1].$$

Thus, we have from (2.16) that

$$\begin{aligned}(\Phi u)(n_0) &= \sum_{s=a}^{b} G(n_0, s)\, f(s, u(w(s)) + u_0(w(s))) \\ &\geq \sigma \sum_{s=a}^{b} G(s,s)\, f(s, u(w(s)) + u_0(w(s))) \\ &\geq \sigma \sum_{E_1} G(s,s)\, f(s, u(w(s))) \\ &\geq \sigma^2 MR \sum_{E_1} G(s,s) \\ &> R = \|u\|,\end{aligned}$$

which leads to

$$\|\Phi u\| > \|u\|, \qquad \forall u \in \partial\mathcal{K}_R.$$

Thus, $i(\Phi, \mathcal{K}_R, \mathcal{K}) = 0$ for $R \geq R_0$. The proof is completed. ∎

In the next theorem, we will also assume the following condition on $f(n,v)$.

(H5) $\lim_{v\to 0^+} \inf \min_{n\in[a,b]} (f(n,v)/v) > k\lambda_1$, $\lim_{v\to+\infty} \sup \max_{n\in[a,b]} (f(n,v)/v) < q\lambda_1$, where $k > 0$ is large enough, such that

$$k\sigma \sum_{E_1} \phi_1(n) \geq \sum_{n=a}^{b} \phi_1(n),$$

and $q > 0$ is small enough, such that

$$\sigma \sum_{n=a}^{b-1} \phi_1(n) \geq q \sum_{n=a}^{b} \phi_1(n),$$

where $\phi_1(n)$ ($\phi_1(n) > 0$, $n \in [a,b]$) is the eigenfunction related to the smallest eigenvalue λ_1 ($\lambda_1 > 0$) of the eigenvalue problem

$$-\Delta^2 \phi_1(n-1) = \lambda \phi_1(n), \qquad \alpha\phi_1(a-1) - \beta\Delta\phi_1(a-1) = 0, \qquad \gamma\phi_1(b) + \delta\Delta\phi_1(b) = 0.$$

THEOREM 2.1. *Assume that Conditions (H1)–(H5) are satisfied. Then, the BVP (1.1) has at least one positive solution.*

PROOF. Fix $0 < m < 1 < m_1$ and let $f_1(u) = u^m + u^{m_1}$ for $u \geq 0$. Then, $f_1(u)$ satisfies (2.14). Define $\Phi_1 : \mathcal{K} \to \mathcal{K}$ by

$$(\Phi_1 u)(n) = \begin{cases} \left(1+\dfrac{\alpha}{\beta}\right)^{n-a} \sum\limits_{s=a}^{b} G(a,s)\, f_1\left(u\left(w\left(s\right)\right) + u_0\left(w\left(s\right)\right)\right), & \tau_1 \leq n \leq a, \\ \sum\limits_{s=a}^{b} G(n,s)\, f_1\left(u\left(w\left(s\right)\right) + u_0\left(w\left(s\right)\right)\right), & a \leq n \leq b, \\ \left(1-\dfrac{\gamma}{\delta}\right)^{n-b} \sum\limits_{s=a}^{b} G(b,s)\, f_1\left(u\left(w\left(s\right)\right) + u_0\left(w\left(s\right)\right)\right), & b \leq n \leq \tau_2. \end{cases} \tag{2.17}$$

Then, by using Lemma 2.5, we conclude that there exist $0 < r_0 < R_0 < \infty$, such that

$$0 < r \leq r_0 \text{ implies } i\left(\Phi_1, \mathcal{K}_r, \mathcal{K}\right) = 0 \tag{2.18}$$

and

$$R \geq R_0 \text{ implies } i\left(\Phi_1, \mathcal{K}_R, \mathcal{K}\right) = 0. \tag{2.19}$$

Define $H : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H(t,u) = (1-t)\Phi u + \Phi_1 u$, then, H is a completely continuous operator. By the first inequality in (H5) and the definition of f_1, there are $\varepsilon > 0$ and $0 < r_1 \leq r_0$, such that

$$\begin{aligned} f(n,u) &\geq (k\lambda_1 + \varepsilon)u, \qquad \forall n \in [a,b], \quad 0 \leq u \leq r_1, \\ f_1(u) &\geq (k\lambda_1 + \varepsilon)u, \qquad \forall 0 \leq u \leq r_1. \end{aligned} \tag{2.20}$$

We now prove that $H(t,u) \neq u$, for all $t \in [0,1]$ and $u \in \partial\mathcal{K}_{r_1}$. In fact, if there exists $t_0 \in [0,1]$ and $u_1 \in \partial\mathcal{K}_{r_1}$, such that $H(t_0,u_1) = u_1$, then, $u_1(n)$ satisfies the equation

$$-\Delta^2 u_1(n-1) = (1-t_0)\, f\left(n, u_1\left(w\left(n\right)\right) + u_0\left(w\left(n\right)\right)\right) + t_0 f_1\left(u_1\left(w\left(n\right)\right) + u_0\left(w\left(n\right)\right)\right), \quad a \leq n \leq b,$$

and the boundary condition. Multiplying the last equation by $\phi_1(n)$ and then, summing it from a to b, using summation by parts in the left-hand side two times, we get that

$$\begin{aligned} \lambda_1 \sum_{n=a}^{b} \phi_1(n) u_1(n) &= \sum_{n=a}^{b} \left[(1-t_0)\, f\left(n, u_1\left(w(n)\right) + u_0\left(w(n)\right)\right) + t_0 f_1\left(u_1\left(w\left(n\right)\right)\right.\right. \\ &\qquad \left.\left. + u_0\left(w\left(n\right)\right)\right)\right] \phi_1\left(n\right) \\ &\geq \sum_{E_1} \left[(1-t_0)\, f\left(n, u_1\left(w\left(n\right)\right)\right) + t_0 f_1\left(u_1\left(w\left(n\right)\right)\right)\right] \phi_1\left(n\right), \end{aligned} \tag{2.21}$$

we obtain from (2.20), that

$$\begin{aligned} &\geq \sum_{E_1} \left[(1-t_0)\left(k\lambda_1 + \varepsilon\right) u_1\left(w\left(n\right)\right) + t_0\left(k\lambda_1 + \varepsilon\right) u_1\left(w\left(n\right)\right)\right] \phi_1\left(n\right) \\ &= \left(\lambda_1 + \frac{\varepsilon}{k}\right) k \sum_{E_1} \phi_1\left(n\right) u_1\left(w\left(n\right)\right) \\ &\geq \left(\lambda_1 + \frac{\varepsilon}{k}\right) k\sigma \left\|u_1\right\| \sum_{E_1} \phi_1\left(n\right) \\ &\geq \left(\lambda_1 + \frac{\varepsilon}{k}\right) \left\|u_1\right\| \sum_{n=a}^{b} \phi_1\left(n\right). \end{aligned} \tag{2.22}$$

We also have

$$\lambda_1 \sum_{n=a}^{b} \phi_1(n) u_1(n) \le \lambda_1 \|u_1\| \sum_{n=a}^{b} \phi_1(n), \tag{2.23}$$

which, together with (2.22), leads to

$$\lambda_1 \ge \lambda_1 + \frac{\varepsilon}{k}.$$

This is impossible. Thus, $H(t,u) \ne u$ for $u \in \partial \mathcal{K}_{r_1}$ and $t \in [0,1]$. By (2.18) and the homotopy invariance of the fixed-point index (see [9]), we get that

$$\begin{aligned} i(\Phi, \mathcal{K}_{r_1}, \mathcal{K}) &= i(H(0,\cdot), \mathcal{K}_{r_1}, \mathcal{K}) \\ &= i(H(1,\cdot), \mathcal{K}_{r_1}, \mathcal{K}) \\ &= i(\Phi_1, \mathcal{K}_{r_1}, \mathcal{K}) \\ &= 0. \end{aligned} \tag{2.24}$$

On the other hand, according to the second inequality of (H5), there exist $\epsilon > 0$ and $R' > R_o$, such that

$$f(n,u) \le (q\lambda_1 - \varepsilon) u, \qquad u > R' \text{ and } n \in [a,b].$$

Set

$$\mathcal{C} = \max_{a \le n \le b,\, 0 \le u \le R'} |f(n,u) - (q\lambda_1 - \varepsilon)u| + 1,$$

then, we deduce that

$$f(n,u) \le (q\lambda_1 - \varepsilon) u + \mathcal{C}, \qquad u \ge 0 \text{ and } n \in [a,b]. \tag{2.25}$$

Define $H_1 : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H_1(t,u) = t\Phi u$, then, H_1 is a completely continuous operator.

We now prove that there exists $R_1 \ge R'$, such that $H_1(t,u) \ne u$, for any $0 \le t \le 1$ and $u \in \mathcal{K}$, $\|u\| \ge R_1$. In fact, if $0 \le t_0 \le 1$ and $u_1 \in \mathcal{K}$ satisfy $H_1(t_0, u_1) = u_1$, then,

$$\begin{aligned} \lambda_1 \sum_{n=a}^{b} u_1(n) \phi_1(n) &\le \sum_{n=a}^{b} f(n, u_1(w(n)) + u_0(w(n))) \phi_1(n) \\ &\le q\left(\lambda_1 - \frac{\varepsilon}{q}\right) \|u_1 + u_0\| \sum_{n=a}^{b} \phi_1(n) + \mathcal{C} \sum_{n=a}^{b} \phi_1(n) \\ &\le q\left(\lambda_1 - \frac{\varepsilon}{q}\right) \|u_1\| \sum_{n=a}^{b} \phi_1(n) + \mathcal{C}_1 \sum_{n=a}^{b} \phi_1(n) \end{aligned} \tag{2.26}$$

and

$$\lambda_1 \sum_{n=a}^{b} u_1(n) \phi_1(n) \ge \lambda_1 \sum_{n=a}^{b-1} u_1(n) \phi_1(n) \ge \lambda_1 \sigma \|u_1\| \sum_{n=a}^{b-1} \phi_1(n) \ge \lambda_1 q \|u_1\| \sum_{n=a}^{b-1} \phi_1(n), \tag{2.27}$$

where $\mathcal{C}_1 = q(\lambda_1 - \varepsilon/q)\|u_0\| + \mathcal{C}$. Combining (2.26) with (2.27), we have

$$\|u_1\| \le \frac{\mathcal{C}_1}{\varepsilon} = \tilde{R}_1.$$

Hence, if

$$R_1 = \max\{R', \tilde{R}_1\} + 1$$

then, we have that

$$H_1(t,u) \ne u, \qquad \text{for } t \in [0,1], \quad u \in \mathcal{K}, \quad \|u\| \ge R_1.$$

Therefore, we have by the homotopy invariance of the fixed-point index,

$$\begin{aligned} i\left(\Phi, \mathcal{K}_{R_1}, \mathcal{K}\right) &= i\left(H_1\left(1, \cdot\right), \mathcal{K}_{R_1}, \mathcal{K}\right) \\ &= i\left(H_1\left(0, \cdot\right), \mathcal{K}_{R_1}, \mathcal{K}\right) \\ &= i\left(\Theta, \mathcal{K}_{R_1}, \mathcal{K}\right) \\ &= 1, \end{aligned} \tag{2.28}$$

where Θ is zero operator. Use (2.24) and (2.28) to conclude that

$$i(\Phi, \mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}, \mathcal{K}) = i\left(\Phi, \mathcal{K}_{R_1}, \mathcal{K}\right) - i\left(\Phi, \mathcal{K}_{r_1}, \mathcal{K}\right) = 1 - 0 = 1.$$

Therefore, Φ has a fixed point in $(\mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1})$.

Let $y(n) = u(n) + u_0(n)$. Since $y(n) = u(n)$ for $n \in [a,b]$ and $0 < r_1 \leq \|u\| = \|u\|_{[a,b]} = \|y\|_{[a,b]} \leq R_1$.

(H6)

$$\lim_{v \to 0^+} \sup \max_{n \in [a,b]} \frac{f(n,v)}{v} < q\lambda_1,$$

$$\lim_{v \to +\infty} \inf \min_{n \in [a,b]} \frac{f(n,v)}{v} > k\lambda_1, \qquad \varsigma(n) \equiv 0, \quad \eta(n) \equiv 0.$$

THEOREM 2.2. *Assume that Conditions (H1)–(H4) and (H6) are satisfied. Then, the BVP (1.1) has at least one positive solution.*

PROOF. Define $H_1 : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H_1(t,u) = t\Phi u$, then, H_1 is a completely continuous operator. By the first inequality in (H6), there exist $\varepsilon > 0$ and $r_1 : 0 < r_1 \leq r_0$, such that

$$f(n,v) \leq (q\lambda_1 - \varepsilon)\, v, \qquad \forall n \in [a,b], \quad 0 \leq v \leq r_1. \tag{2.29}$$

We now prove that $H_1(t,u) \neq u$, for $0 \leq t \leq 1$ and $u \in \partial\mathcal{K}_{r_1}$. In fact, if there exists $0 \leq t_0 \leq 1$ and $u_1 \in \partial\mathcal{K}_{r_1}$, such that $H_1(t_0,u_1) = u_1$, then, the $u_1(n)$ satisfy the boundary condition. Since $\varsigma(n) \equiv 0$, $\eta(n) \equiv 0$, we have $u_0(n) \equiv 0$.

$$-\Delta^2 u_1(n-1) = t_0 f\left(n, u_1\left(w\left(n\right)\right)\right), \qquad \forall n \in [a,b].$$

Multiplying the last equality by $\phi_1(n)$, and summing from a to b, we see that

$$\lambda_1 \sum_{n=a}^{b} u_1(n)\,\phi_1(n) = t_0 \sum_{n=a}^{b} f\left(n, u_1\left(w\left(n\right)\right)\right)\phi_1(n) \tag{2.30}$$

$$\leq \sum_{n=a}^{b} f\left(n, u_1\left(w\left(n\right)\right)\right)\phi_1(n)$$

$$\leq (q\lambda_1 - \varepsilon)\, \|u_1\| \sum_{n=a}^{b} \phi_1(n) \tag{2.31}$$

and

$$\lambda_1 \sum_{n=a}^{b} u_1(n)\,\phi_1(n) \geq \lambda_1 \sum_{n=a}^{b-1} u_1(n)\,\phi_1(n) \geq \lambda_1 \sigma \|u_1\| \sum_{n=a}^{b-1} \phi_1(n) \geq \lambda_1 q \|u_1\| \sum_{n=a}^{b} \phi_1(n), \tag{2.32}$$

which, together with (2.31), leads to

$$\lambda_1 q \leq \lambda q_1 - \varepsilon.$$

This is impossible. Using homotopy invariance of the fixed-point index, we have that

$$\begin{aligned} i(\Phi, \mathcal{K}_{r_1}, \mathcal{K}) &= i(H_1(0,\cdot), \mathcal{K}_{r_1}, \mathcal{K}) \\ &= i(\Theta, \mathcal{K}_{r_1}, \mathcal{K}) \\ &= 1. \end{aligned} \tag{2.33}$$

Define $H : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H(t,u) = (1-t)\Phi u + t\Phi_1 u$, then, H is a completely continuous operator. By the second inequality in (H6) and the definition of f_1, there exist $\varepsilon > 0$ and $R' > R_0$, such that

$$\begin{aligned} f(n,u) &\geq (k\lambda_1 + \varepsilon) u, \qquad \forall n \in [a,b], \quad u \geq R', \\ f_1(u) &\geq (k\lambda_1 + \varepsilon) u, \qquad \forall u \geq R'. \end{aligned}$$

Let

$$C = \max_{0 \leq u \leq R', n \in [a,b]} |f(n,u) - (k\lambda_1 + \varepsilon) u| + \max_{0 \leq u \leq R'} |f_1(u) - (k\lambda_1 + \varepsilon) u| + 1,$$

then, it is obvious that

$$f(n,u) \geq (k\lambda_1 + \varepsilon) u - c, \qquad \forall n \in [a,b], u \geq 0, \tag{2.34}$$

$$f_1(u) \geq (k\lambda_1 + \varepsilon) u - c, \qquad \forall u \geq 0. \tag{2.35}$$

We now prove that there exists $R_1 \geq R'$, such that $H(t,u) \neq u$ for any $0 \leq t \leq 1$ and $u \in \mathcal{K}$, $\|u\| \geq R_1$. In fact, if $0 \leq t_0 \leq 1$ and $u_1 \in \mathcal{K}$, satisfying $H(t_0, u_1) = u_1$, then, using (2.34) and (2.35), it is analogous to the argument of (2.22) and (2.23) that

$$\begin{aligned} \lambda_1 \sum_{n=a}^{b} \phi_1(n) u_1(n) &= \sum_{n=a}^{b} [(1-t_0) f(n, u_1(w(n))) + t_0 f_1(u_1(w(n)))] \phi_1(n) \\ &\geq \sum_{E_1} [(1-t_0) f(n, u_1(w(n))) + t_0 f_1(u_1(w(n)))] \phi_1(n) \\ &\geq \sum_{E_1} \{(1-t_0) [(k\lambda_1 + \varepsilon) u_1(w(n)) - c] \\ &\quad + t_0 [(k\lambda_1 + \varepsilon) u_1(w(n)) - c]\} \phi_1(n) \\ &= \sum_{E_1} [(k\lambda_1 + \varepsilon) u_1(w(n)) - c] \phi_1(n) \\ &\geq \left(\lambda_1 + \frac{\varepsilon}{k}\right) k\sigma \|u_1\| \sum_{E_1} \phi_1(n) - c \sum_{E_1} \phi_1(n), \end{aligned} \tag{2.36}$$

$$\begin{aligned} \lambda_1 \sum_{n=a}^{b} \phi_1(n) u_1(n) &\leq \lambda_1 \|u_1\| \sum_{n=a}^{b} \phi_1(n) \\ &\leq \lambda_1 \|u_1\| k\sigma \sum_{E_1} \phi_1(n), \end{aligned} \tag{2.37}$$

(2.36) and (2.37) lead to $\|u_1\| \leq c/\varepsilon\sigma = \tilde{R}_1$. Let $R_1 = \max\{R', \tilde{R}_1\} + 1$.

We obtain

$$\begin{aligned} i(\Phi, \mathcal{K}_{R_1}, \mathcal{K}) &= i(H(0,\cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(H(1,\cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(\Phi_1, \mathcal{K}_{R_1}, \mathcal{K}) \\ &= 0. \end{aligned} \tag{2.38}$$

Use (2.33) and (2.38) to conclude that

$$i(\Phi, \mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}, \mathcal{K}) = -1.$$

Therefore, Φ has a fixed point in $\mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}$.

COROLLARY. *Using the following (H7) or (H8) instead of (H5) or (H6), the conclusions of Theorems 2.1 and 2.2 are true. For all* $n \in [a, b]$,

(H7) $\lim_{v\to 0^+}(f(n,v)/v) = +\infty$, $\lim_{v\to+\infty}(f(n,v)/v) = 0$ *(sublinear);*

(H8) $\lim_{v\to 0^+}(f(n,v)/v) = 0$, $\lim_{v\to+\infty}(f(n,v)/v) = +\infty$ *(superlinear),* $\varsigma(n) \equiv 0$, $\eta(n) \equiv 0$.

EXAMPLE. Let us introduce an example to illustrate the usage of our theorems.

Consider the BVP:

$$\begin{aligned} -\Delta^2 y(n-1) &= a(n)\, e^{\sigma y(n-3)}, && 1 \le n \le 7, \\ y(n-1) - \Delta y(n-1) &= e^n, && -2 \le n \le 1, \\ \Delta y(7) &= 0. && \end{aligned} \tag{2.39}$$

It is assumed that $\sigma < 0$ and $a(n)$ is nonnegative on [1,7] and is not identically zero on E_1.

Then, $w(n) = n - 3$, $\tau_1 = -2$, $\tau_2 = 7$, $f(n,v) = a(n)e^{\sigma v}$, $\alpha = \beta = \delta = 1$, $\gamma = 0$.

Since $\lim_{v\to 0^+}(f(n,v)/v) = +\infty$, $\lim_{v\to+\infty}(f(n,v)/v) = 0$. Thus, (H1)–(H4) and corollary are satisfied and (2.39) has at least one positive solution $y(n)$.

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